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Coupling three-field formulation and meshless mixed Galerkin methods using radial basis functions

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ABSTRACT

In this work, we solve the elliptic partial differential equation by coupling the meshless mixed Galerkin approximation using radial basis function with the three-field domain decomposition method. The formulation has been adopted to increase the efficiency of the numerical technique by decreasing the error and dealing with the ill conditioning of the linear system caused by the radial basis function. Convergence analysis of the coupled technique is treated and numerical results of some solved examples are given at the end of this paper.

1. Introduction

The use of a mesh is a basic characteristic of traditional numerical approaches for the solution of partial differential equations (PDEs), such as finite element method (FEM), finite difference method (FDM) and boundary element method (BEM). In those approaches, assumptions are made for the local approximation of the primitive functions, which require mesh to support them. The generation of a mesh is a complicated work. During the last decade, considerable effort has been given to the development of the so-called meshless or meshfree methods. The aim of this type of approach is to eliminate at least the structure of the mesh and approximate the solution entirely using nodal values inside and on the boundary distributed quasi-randomly in the domain. Among the famous meshless methods we can mention the element free Galerkin (EFG) method given in [1], the local Petrov–Galerkin and generalized finite element methods respectively developed in [2,3]. The method of fundamental solution is also considered as one of the important meshless methods.

In recent years, the theory of radial basis functions (RBFs) has undergone intensive research and enjoyed considerable success as a technique for interpolating multivariable data and functions [4]. Although most work to date on RBFs is related to scattered data approximation and in general to interpolation theory, there has recently been an increasing interest in their application for solving PDEs. The approach is based on approximating the whole solution of the PDE by a translation of RBFs. It is very attractive due to the fact that it is a truly meshless and spatial dimension independent method, which can be easily extended to solving high dimensional problems. Furthermore, since the RBFs are smooth, it can be easily applied for solution of high order differential equations. Kansa [5] was the first to propose the use of RBFs to solve PDEs using collocation method. Then, it is extensively studied in [6,7]. Wendland [8] derived error estimates for the solution of the Helmholtz equation with the Neumann boundary condition when meshless Galerkin methods using radial basis functions are employed. Cheng et al. [9] introduced the h–c meshless scheme for smooth problems. Hu et al. [10] derived inverse estimates and error analysis for Multi-quadrics-based collocation method by treating it as the Ritz–Galerkin method. One

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of the most advantages of the use of radial basis functions is the high accuracy of the numerical solutions. But the large condition number of the resulting algebraic system is one of the main drawbacks of their use.

The major technique applied to deal with the large condition number when using radial basis functions, is the domain decomposition method. It is nowadays considered as one of the most popular techniques that can be applied for numerical solution and in the same time be suitable for parallel computation. It has been coupled both with mesh [11] and meshless methods [12]. Hon et al. [12] have shown that using domain decomposition, meshless method based on RBFs, can solve a problem with a large number of nodes.

The idea behind the domain decomposition is to divide the considered domain into a number of sub-domains and then try to solve the original problem by a series of sub-problems that interact through artificial interfaces inside the computational domain. The numerical solution can be computed either iteratively, using Schwartz method [11,12], by exchanging data on interfaces between sub-problems or directly, computing the interface's data with Steklov technique and then using interface's solution to solve each sub-problem separately as in [11–13]. There exist two different approaches for domain decomposition: overlapping and non-overlapping domains.

The main objective of this paper is to couple the three-field domain decomposition method with mixed Galerkin formulation based on radial basis functions to solve Poisson equation [14]. The technique used is based on the Lagrange multipliers method for domain decomposition and penalty technique to impose the Dirichlet boundary condition in the weak formulation. The proposed coupled technique is formulated to increase the efficiency of the mixed Galerkin method using RBFs and also to establish an equilibrium relationship between the error and the stability of the algebraic system. The goal can be achieved by keeping the condition number around numerically accepted value and decreasing the approximation error for a large number of distributed source points.

An outline of the paper is as follows. In Section 2, we recall the three-field domain decomposition method. In Section 3, we give the radial basis functions formulation and discuss the validity of the two *inf–sup* conditions as well as the convergence theorem. Numerical results of some solved examples are given in Section 4 of the paper.

2. Three-field domain decomposition formulation

2.1. Continuous formulation

Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or $n = 3$) be a bounded polygonal spatial open domain of Lipschitz boundary $\partial\Omega$ and consider the following partial differential equation:

$$(P) \quad \begin{cases} -\mathrm{div}\,(K(x)\nabla u(x)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where K is a positive definite matrix and $f \in L^2(\Omega)$. Assuming some regularity on K and f, the problem (1) has a unique solution.

Let us split Ω into a set of non-overlapping sub-domains Ω_k, $k = 1, \ldots, m$ such that

$$\overline{\Omega} = \bigcup_{k=1}^{m} \overline{\Omega_k}, \qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j \tag{2}$$

and put

$$\Sigma = \bigcup_{k=1}^{m} \Gamma^k \quad \text{where } \Gamma^k = \partial\Omega_k \setminus \partial\Omega. \tag{3}$$

Let V be the product function space given as:

$$V = \prod_{k=1}^{m} H^1(\Omega_k) \tag{4}$$

which is isomorphic to

$$V = \{v \in L^2(\Omega) \mid v^k = v|_{\Omega_k} \in H^1(\Omega)\} \tag{5}$$

Λ and Φ are defined by:

$$\Lambda = \prod_{k=1}^{m} H^{-\frac{1}{2}}(\partial\Omega_k) \tag{6}$$

$$\Phi = \{\varphi \in L^2(\Sigma) \mid \exists\, u \in H_0^1(\Omega),\ u = \varphi \text{ on } \Sigma\} = H_0^1(\Omega)|_{\Sigma}.$$

The three function spaces are, respectively, endowed with the following norms

$$\|u\|_V^2 = \sum_{k=1}^m \|u^k\|_{H^1(\Omega_k)}^2, \qquad \|\lambda\|_\Lambda^2 = \sum_{k=1}^m \|\lambda^k\|_{H^{-\frac{1}{2}}(\partial\Omega_k)}^2$$

and

$$\|\varphi\|_\Phi = \inf_{u=\varphi \text{ on } \Sigma} \|u\|_{H^1(\Omega)} \simeq \left(\sum_{k=1}^m |\varphi^k|_{H^{\frac{1}{2}}(\Gamma^k)}^2\right)^{\frac{1}{2}}$$

where $|.|_{H^{\frac{1}{2}}(\Gamma^k)}$ is defined in [11].

The three-field domain decomposition formulation of the problem (1) is as follow [11]:

Find $(u, \lambda, \varphi) \in V \times \Lambda \times \Phi$ such that $\forall k = 1, \ldots, m$

$$\begin{cases} \forall v^k \in H^1(\Omega_k), \qquad \forall \mu^k \in H^{-\frac{1}{2}}(\partial\Omega_k) \\ a^k(u^k, v^k) + b^k(v^k, \lambda^k) = \langle f, v^k\rangle_{L^2(\Omega_k)} \\ b^k(u^k, \mu^k) + c^k(\mu^k, \varphi^k) = 0 \\ \text{and} \quad \forall\, \psi \in \Phi, \qquad \displaystyle\sum_{k=1}^m c^k(\lambda^k, \psi) = 0 \end{cases} \tag{7}$$

where u^k, λ^k and φ^k are respectively the restrictions of u, λ and φ to Ω_k, $\partial\Omega_k$ and $\partial\Omega_k \setminus \partial\Omega$, which means that $u^k = u|_{\Omega_k}$, $\lambda^k = \lambda|_{\partial\Omega_k}$ and $\varphi^k = \varphi|_{\Gamma^k}$.

The bilinear forms a^k, b^k and c^k, given in the formulation (7), are defined as

$$a^k(v, w) = \int_{\Omega_k} \left(\sum_{i,j=1}^2 K_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}\right) \mathrm{d}x, \quad \forall (u, w) \in H^1(\Omega_k) \times H^1(\Omega_k),$$

$$b^k(v, \lambda) = \int_{\partial\Omega_k} v\lambda \mathrm{d}s, \quad \forall\, (v, \lambda) \in H^1(\Omega_k) \times H^{-\frac{1}{2}}(\partial\Omega_k)$$

and

$$c^k(\lambda, \varphi) = \int_{\partial\Omega_k} \lambda\varphi \mathrm{d}s \quad \forall\, (\lambda, \varphi) \in H^{-\frac{1}{2}}(\partial\Omega_k) \times \Phi.$$

As demonstrated in [15] for all $f \in L^2(\Omega)$ the problem (7) has a unique solution (u, λ, φ) that satisfies:

$$\begin{cases} u^k = u^* & \text{in } \Omega_k \ \forall\, k = 1, \ldots, m \\ \lambda^k = K \dfrac{\partial u^*}{\partial n_k} & \text{on } \partial\Omega_k \forall k = 1, \ldots, m \\ \varphi = u^* & \text{on } \Sigma \end{cases}$$

where $\frac{\partial u^*}{\partial n_k}$ is the outward normal derivative of u^* on $\partial\Omega_k$ and u^* is the solution of the problem (1).

2.2. Discretization spaces and algebraic formulation

Choosing three finite dimensional function spaces $V_h = \prod_{k=1}^m V_h^k \subset V$, $\Lambda_h = \prod_{k=1}^m \Lambda_h^k \subset \Lambda$ and $\Phi_h \subset \Phi$, where V_h^k and Λ_h^k are the finite dimensional spaces approximating, respectively, $H^1(\Omega_k)$ and $H^{-\frac{1}{2}}(\partial\Omega_k)$, we have the following approximated problem

Find $(u_h, \lambda_h, \varphi_h) \in V_h \times \Lambda_h \times \Phi_h$ such that $\forall k = 1, \ldots, m$

$$\begin{cases} \forall v_h^k \in V_h^k(\Omega_k), \quad \forall \mu_h^k \in \Lambda_h^k \quad \text{and} \quad \forall\, \psi_h \in \Phi_h \\ a^k(u_h^k, v_h^k) + b^k(v_h^k, \lambda_h^k) = \langle f, v_h^k\rangle_{L^2(\Omega_k)} \\ b^k(u_h^k, \mu_h^k) + c^k(\mu_h^k, \varphi_h) = 0 \\ \displaystyle\sum_{k=1}^m c^k(\lambda_h^k, \psi_h) = 0. \end{cases} \tag{8}$$

Based on the validation of the two *inf*–*sup* conditions

- $\inf_{\lambda_h\in\Lambda_h}\sup_{v_h\in V_h}\frac{\sum_{k=1}^{m} b^k(v_h^k,\lambda_h^k)}{\|\lambda_h\|_\Lambda\|v_h\|_V}\geq\beta_1\geq 0$
- $\inf_{\varphi_h\in\Phi_h}\sup_{\lambda_h\in\Lambda_h}\frac{\sum_{k=1}^{m} c^k(\lambda_h^k,\varphi_h)}{\|\varphi_h\|_\Phi\|\lambda_h\|_\Lambda}\geq\beta_2\geq 0$

for the constructed function spaces, we have the existence, uniqueness and stability of the solution of the discretized system and we can also drive the following estimation of the error [11]

$$\|u-u_h\|_V+\|\lambda-\lambda_h\|_\Lambda+\|\varphi-\varphi_h\|_\Phi\leq C[\inf_{v_h\in V_h}\|u-v_h\|_V+\inf_{\mu_h\in\Lambda_h}\|\lambda-\mu_h\|_\Lambda+\inf_{\psi_h\in\Phi_h}\|\varphi-\psi_h\|_\Phi]. \tag{9}$$

Letting α_k and γ_k be the components of u_h^k and λ_h^k, respectively, the system (8) can be transformed into an algebraic one in the following form:

$$\begin{pmatrix} A_1 & B_1^T & 0 & 0 & 0 & 0 & 0\\ B_1 & 0 & 0 & 0 & 0 & 0 & C_1^T\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & A_k & B_k^T & 0 & 0 & 0\\ 0 & 0 & B_k & 0 & 0 & 0 & C_k^T\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & A_m & B_m^T & 0\\ 0 & 0 & 0 & 0 & B_m^T & 0 & C_m^T\\ 0 & C_1 & 0 & C_k & 0 & C_m & 0 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \gamma_1\\ \cdots\\ \alpha_k\\ \gamma_k\\ \cdots\\ \alpha_m\\ \gamma_m\\ \varphi\end{pmatrix}=\begin{pmatrix} f_1\\ 0\\ \cdots\\ f_k\\ 0\\ \cdots\\ f_m\\ 0\\ 0\end{pmatrix}. \tag{10}$$

We can observe that the matrix is sparse and if the φ is known, then the algebraic system in each sub-domain Ω_k written as

$$\begin{pmatrix} A_k & B_k^T & 0\\ B_k & 0 & C_k^T\\ 0 & C_k^T & 0\end{pmatrix}\begin{pmatrix}\alpha_k\\ \gamma_k\\ \varphi\end{pmatrix}=\begin{pmatrix} f_k\\ 0\\ 0\end{pmatrix} \tag{11}$$

will be reduced into

$$\begin{pmatrix} A_k & B_k^T\\ B_k & 0\end{pmatrix}\begin{pmatrix}\alpha_k\\ \gamma_k\end{pmatrix}=\begin{pmatrix} f_k\\ -C_k^T\varphi\end{pmatrix} \tag{12}$$

and then we have m independent systems to solve in parallel.

3. Convergence analysis

For all $k=1,\ldots,m$, let X_1^k and X_2^k be two sets of finite number of distinct centers given by

$$X_1^k=\{x_1,x_2,\ldots,x_{N_k}\}\subset\overline{\Omega_k}\quad\text{and}\quad X_2^k=\{x_1,x_2,\ldots,x_{M_k}\}\subset\partial\Omega_k$$

and put

$$h_1^k=\sup_{x\in\Omega_k}\inf_{x_j\in X_1^k}\|x-x_j\|\quad\text{and}\quad h_2^k=\sup_{x\in\partial\Omega_k}\inf_{x_j\in X_2^k}\|x-x_j\|.$$

We define the approximated function spaces as

$$\forall k=1,\ldots,m,\quad V_h^k=\operatorname{span}\{\phi(\|.-x_j\|),x_j\in X_1^k\},$$
$$\forall k=1,\ldots,m,\quad \Lambda_h^k=\operatorname{span}\{\psi(\|.-x_j\|),x_j\in X_2^k\}.$$

One can choose the radial basis functions ϕ and ψ such that $V_h^k\subset H^1(\Omega_k)$, $\Lambda_h^k\subset H^{-\frac{1}{2}}(\partial\Omega_k)$ and $\bar{V}_h^k=H^1(\Omega_k)$, $\bar{\Lambda}_h^k=H^{-\frac{1}{2}}(\partial\Omega_k)$. For the approximated space Φ_h we use the characteristic function w_j^k defined by

$$w_j^k=\begin{cases}1 & \text{if } x\in\Gamma_j^k=\Gamma^k\cap\Gamma^j\\ 0 & \text{if not.}\end{cases}$$

Then, we define the function space Φ_h by:

$$\Phi_h=\operatorname{span}\{w_j^k\phi(\|.-x_p\|):x_p\in X_3\cap\Gamma_j^k,\ 1\leq k\leq m\text{ and }1\leq j\leq m\}$$

where X_3 is a set of all interiors points of Ω that belong to Σ.

After the construction of approximated function spaces we have to verify the two *inf–sup* conditions. For that we need to recall the following theorem.

Theorem 3.1 (*[8]*). *Let $\Omega \in \mathbb{R}^d$ be an open and bounded domain, having Lipschitz boundary and ϕ a positive definite radial basis function such that its Fourier transform satisfies $\widehat{\phi} \sim (1 + \|\omega\|_2)^{-2l}$. Denote by $s_u = \sum_{i=1}^{N} \alpha_i \phi(\|. - x_i\|)$ the interpolant on $X = \{x_1, x_2, \ldots, x_N\} \subseteq \Omega$ to $u \in H^k(\Omega)$ with $k > d/2$. Then there exists a constant $h_0 > 0$ such that for all X with $h < h_0$ where h is the density of X, the estimate $\|u - s_u\|_{H^j(\Omega)} \le Ch^{l-j}\|u\|_{H^l(\Omega)}$ is valid for $0 \le j \le l$.*

Let us then demonstrate the two *inf–sup* conditions.

Lemma 3.2. *Let ϕ and ψ be two positive definite radial basis functions and assume that their Fourier transforms satisfy $\widehat{\phi}(t) \sim (1 + \|t\|)^{-l-d}$ and $\widehat{\psi}(t) \sim (1 + \|t\|)^{-l-d}$. If $h_1^k(h_2^k)^{-2l-3d+3}$ is very small for all $k = 1, \ldots, m$, then $\exists \beta_1 > 0$ such that*

$$\inf_{\lambda_h \in \Lambda_h} \sup_{v_h \in V_h} \frac{\sum_{k=1}^{m} b^k(v_h^k, \lambda_h^k)}{\|\lambda_h\|_\Lambda \|v_h\|_V} \ge \beta_1.$$

Proof. Let $\lambda_h = (\lambda_h^k)_{k=1}^m \in \Lambda_h$. Following [16] and for $h_1^k(h_2^k)^{-2l-3d+3}$ very small, there exists $\beta_1^k > 0$ such that

$$\forall \lambda_h^k \in \Lambda_h^k \, \exists v_h^k \in V_h^k : \|v_h^k\|_{V^k} = 1, \quad \text{and} \quad b^k(v_h^k, \lambda_h^k) \ge \beta_1^k \|\lambda_h^k\|_{\Lambda^k}$$

so $\forall \lambda_h \in \Lambda_h \;\; \exists v_h = \frac{1}{m}(v_h^k)_{k=1}^m \in V_h : \|v_h\|_V = 1,$ and $\sum_{k=1}^m b^k(v_h^k, \lambda_h^k) \ge \beta_1 \|\lambda_h\|_\Lambda$ with $\beta_1 = \inf_{k=1,\ldots,m}\{\beta_1^k\}$. □

For the demonstration of the second *inf–sup* condition we need the following lemma.

Lemma 3.3. *If the Fourier transform of the positive definite radial basis function φ that span the functional space Φ_h satisfies $\widehat{\varphi(t)} \sim (1 + \|t\|)^{-l-d}$, then for all $g \in \Phi_h$ one has*

$$\|g\|_{H^{\frac{3}{2}}(\partial\Omega_k)} \le C_k (h_3^k)^{-l-\frac{3}{2}d+\frac{3}{2}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega_k)} \quad \forall k = 1, \ldots, m. \tag{13}$$

where $X_3^{kj} = X_3 \cap \Gamma_j^k$, $h_3^k = \inf_j h_3^{kj}$ and $h_3^{kj} = \sup_{x \in \Gamma_j^k} \inf_{x_j \in X_3^{kj}} \|x - x_j\|$.

Proof. Let $I = \{(j, k)$ such that $\Gamma^k \bigcap \Gamma^j \neq \emptyset\}$ and $g \in \Phi_h$, then

$$g = \sum_{(j,k)\in I} w_j^k g_j^k \quad \text{where } g_j^k \in \mathrm{span}\{\varphi(\|. - x_i\|) : x_i \in X_3^{kj}\}.$$

Putting $g_j^k = \sum_{p=1}^{N_{kj}} \alpha_p^{kj} \varphi(\|. - x_p\|)$, where N_{kj} is the number of nodes in X_3^{kj} and following [16], we have

$$\|g_j^k\|^2_{L^2(\Gamma_j^k)} \ge c_j^k (h_3^{kj})^{2l+2d-2} \|\alpha^{kj}\|^2, \quad \text{where } \|\alpha^{kj}\|^2 = \sum_{p=1}^{N_{kj}} |\alpha_p^{kj}|^2$$

and

$$\begin{aligned}
\|g_j^k\|^2_{H^2(\Omega_k)} &\le c_j^k N_{kj} \max_p \|\varphi(\|. - x_p\|)\|^2_{H^2(\Omega_k)} \|\alpha^{kj}\|^2 \\
&\le C_j^k N_{kj} \|\alpha^{kj}\|^2 \\
&\le C_j^k (h_3^{kj})^{-d+1} \|\alpha^{kj}\|^2
\end{aligned}$$

then we have

$$\begin{aligned}
\|g\|^2_{H^2(\Omega_k)} = \left\| \sum_{(j,l)\in I} w_j^l g_j^l \right\|^2_{H^2(\Omega_k)} &\le \sum_j \|g_j^k\|^2_{H^2(\Omega_k)} \\
&\le \sum_j C_j^k (h_3^{kj})^{-d+1} \|\alpha^{kj}\|^2 \\
&\le C_{max}^k \sum_j (h_3^{kj})^{-d+1} \|\alpha^{kj}\|^2
\end{aligned}$$

where $C^k_{max} = \max_j C^k_j$, so

$$\begin{aligned}
\|g\|^2_{L^2(\partial\Omega_k)} = \sum_j \|g^k_j\|^2_{L^2(\Gamma^k_j)} &\geq \sum_j c^k_j (h^{kj}_3)^{2l+2d-2}\|\alpha^{kj}\|^2 \\
&\geq C^k_{min}\sum_j (h^{kj}_3)^{2l+2d-2}\|\alpha^{kj}\|^2 \\
&\geq C^k_{min}\sum_j (h^{kj}_3)^{2l+3d-3}(h^{kj}_3)^{-d+1}\|\alpha^{kj}\|^2 \\
&\geq C^k_{min}(h^k_3)^{2l+3d-3}\sum_j (h^{kj}_3)^{-d+1}\|\alpha^{kj}\|^2 \\
&\geq C_k (h^k_3)^{2l+3d-3}\|g\|^2_{H^2(\Omega_k)}
\end{aligned}$$

where $C^k_{min} = \min_j C^k_j$ and $C_k = \frac{C^k_{min}}{C^k_{max}}$.

As

$$\begin{aligned}
\|g\|_{H^{\frac{3}{2}}(\partial\Omega_k)} &= \inf_{h\in H^2(\Omega_k),\, g=h|_{\partial\Omega_k}} \|h\|_{H^2(\Omega_k)} \leq \|g\|_{H^2(\Omega_k)} \\
&\leq C_k (h^k_3)^{-l-\frac{3}{2}d+\frac{3}{2}}\|g\|_{L^2(\partial\Omega_k)}
\end{aligned}$$

and as $\|g\|_{L^2(\partial\Omega_k)} \leq \|g\|_{H^{\frac{1}{2}}(\partial\Omega_k)}$ thus, the lemma is proven. □

After this lemma we verify the second condition of *inf–sup*.

Lemma 3.4. *Let ψ and φ be two positive definite radial basis functions and assume that their Fourier transforms satisfy $\widehat{\psi}(t) \sim (1+\|t\|)^{-l-d}$ and $\widehat{\varphi}(t) \sim (1+\|t\|)^{-l-d}$. If $(h_k)^2(h^k_3)^{-l-\frac{3}{2}d+\frac{3}{2}}$ and $h^k_1(h^k_2)^{-2l-3d+3}$ are small for all $k = 1, \ldots, m$, where $h_k = \max\{h^k_1, h^k_2\}$, then there exists $\beta_2 > 0$ such that*

$$\inf_{\varphi_h\in\Phi_h}\sup_{\lambda_h\in\Lambda_h}\frac{\sum_{k=1}^m c^k(\lambda^k_h, \varphi_h)}{\|\varphi_h\|_\Phi\|\lambda_h\|_\Lambda} \geq \beta_2.$$

Proof. Letting $\varphi_h \in \Phi_h$, one has $\varphi^k_h = \varphi_h|_{\partial\Omega_k} \in H^{\frac{1}{2}}(\partial\Omega_k)$.

As in [17], there exists $w^k \in H^1(\Omega_k)$ such that $\|\varphi^k_h\|_{H^{\frac{1}{2}}(\partial\Omega_k)} = \|w^k\|_{H^1(\Omega_k)}$ and w^k solves the following problem

$$\begin{cases} -\Delta w^k + w^k = 0 & \text{in } \Omega_k \\ w^k = \varphi^k_h & \text{on } \partial\Omega_k \end{cases}$$

and as in [16] if radial basis functions that span discretized approximation function spaces Λ^k_h and V^k_h satisfy the condition of the lemma and if $h^k_1(h^k_2)^{-2l-3d+3}$ is small there exist $(w^k_h, \lambda^k_h) \in V^k_h \times \Lambda^k_h$ such that we have

$$\left\|\frac{\partial w^k}{\partial n} - \lambda^k_h\right\|_{H^{-\frac{1}{2}}(\partial\Omega_k)} + \|w^k - w^k_h\|_{H^1(\Omega_k)} \leq C_k (h_k)^2 \|\varphi^k_h\|_{H^{\frac{3}{2}}(\partial\Omega_k)}$$

then we also have

$$\|w^k\|_{H^1(\Omega_k)} \leq \|w^k - w^k_h\|_{H^1(\Omega_k)} + \|w^k_h\|_{H^1(\Omega_k)} \leq C_k(h_k)^2\|\varphi^k_h\|_{H^{\frac{3}{2}}(\partial\Omega_k)} + \|w^k_h\|_{H^1(\Omega_k)}$$

and if we use the Lemma 3.3 we obtain

$$\|w^k\|_{H^1(\Omega_k)} \leq \|w^k - w^k_h\|_{H^1(\Omega_k)} + \|w^k_h\|_{H^1(\Omega_k)} \leq C_k(h_k)^2(h^k_3)^{-l-\frac{3}{2}d+\frac{3}{2}}\|w^k\|_{H^{\frac{1}{2}}(\partial\Omega_k)} + \|w^k_h\|_{H^1(\Omega_k)} \tag{14}$$

which can give us the following inequality

$$(1 - C_k(h_k)^2(h^k_3)^{-l-\frac{3}{2}d+\frac{3}{2}})\|w^k\|_{H^1(\Omega_k)} \leq \|w^k_h\|_{H^1(\Omega_k)}.$$

Using the mixed Galerkin formulation in [16]

$$\begin{cases} \int_{\Omega_k} \nabla w^k_h \nabla v^k_h + w^k_h v^k_h + \int_{\partial\Omega_k} \lambda^k_h \varphi^k_h \mathrm{d}s = 0 & \forall v^k_h \in V^k_h \\ \int_{\partial\Omega_k} w^k_h \mu^k_h = \int_{\partial\Omega_k} \varphi^k_h \mu^k_h & \forall \mu^k_h \in \Lambda^k_h \end{cases}$$

and choosing $v_h^k = w_h^k$ and $\mu_h^k = \lambda_h^k$, we have

$$\int_{\partial\Omega_k} (-\lambda_h^k)\varphi_h^k \mathrm{d}s = \|w_h^k\|^2_{H^1(\Omega_k)}. \tag{15}$$

Then, we get

$$\begin{aligned}\int_{\partial\Omega_k} (-\lambda_h^k)\varphi_h^k \mathrm{d}s &\geq (1 - C_k(h_k^2)^2(h_3^k)^{-l-\frac{3}{2}d+\frac{3}{2}})\|w^k\|^2_{H^1(\Omega_k)} \\ &\geq (1 - C_k(h_k)^2(h_3^k)^{-l-\frac{3}{2}d+\frac{3}{2}})\|\varphi_h^k\|^2_{H^{\frac{1}{2}}(\partial\Omega_k)}\end{aligned} \tag{16}$$

$$\int_{\partial\Omega_k} (-\lambda_h^k)\varphi_h^k \mathrm{d}s \geq \beta_2^k \|\varphi_h\|_{H^{\frac{1}{2}}(\partial\Omega_k)} \|\lambda_h^k\|_{H^{-\frac{1}{2}}(\partial\Omega_k)}. \tag{17}$$

Then

$$\forall \varphi_h \in \Phi_h \text{ and } \forall k = 1, \dots, m, \quad \exists \lambda_h^k \text{ such that } c^k(-\lambda_h^k, \varphi_h) \geq \beta_2^k \|\varphi_h\|_{H^{\frac{1}{2}}(\partial\Omega_k)} \|\lambda_h^k\|_{H^{-\frac{1}{2}}(\partial\Omega_k)}.$$

If we put $\lambda_h = \frac{1}{m}(\lambda_h^k)$ and $\beta_2 = \inf_k \beta_2^k$ then we have

$$\begin{aligned}\sum_{k=1}^m c^k\left(-\frac{\lambda_h^k}{\|\lambda_h^k\|_{\Lambda_h^k}}, \varphi_h\right) &\geq \beta_2 \sum_{k=1}^m \|\varphi^k\|_{H^{\frac{1}{2}}(\partial\Omega_k)} \\ &\geq c\beta_2 \sum_{k=1}^m |\varphi^k|_{H^{\frac{1}{2}}(\partial\Omega_k)}.\end{aligned}$$

Putting $\mu_h = (\mu_h^k)_{k=1}^m = \frac{1}{m}\left(-\frac{\lambda_h^k}{\|\lambda_h^k\|_{\Lambda_h^k}}\right)_{k=1}^m$ which gives $\|\mu_h\|_\Lambda = 1$ and as $\|\varphi\|_\Phi \sim \sum_{k=1}^m |\varphi^k|_{H^{\frac{1}{2}}(\partial\Omega_k)}$ then we have,

$$\sum_{k=1}^m c^k(\mu_h^k, \varphi_h) \geq c\beta_2 \|\varphi\|_\Phi.$$

So, one obtains the second condition *inf–sup*. □

Theorem 3.5. *Let $f \in L^l(\Omega)$, $l \geq 0$ and u be the solution of problem (P). If $(u_h, \lambda_h, \varphi_h)$ is the approximated solution using 3-FFRBF such that $(h_k)^2(h_3^k)^{-l-\frac{3}{2}d+\frac{3}{2}}$ and $(h_1^k)(h_2^k)^{-2l-3d+3}$ are sufficiently small then one has*

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_\Lambda + \|\varphi - \varphi_h\|_\Phi \leq C\left(h_1^{(l-1)}\|u\|_{H^l(\Omega)} + h_2^{s+\frac{3}{2}}\|\lambda\|_{H^{s+1}(\partial\Omega)} + h_3^{s+\frac{3}{2}}\|\varphi\|_{H^{s+1}(\partial\Omega)}\right)$$

for $0 < s + 1 < l$, where $h_1 = \max_k\{h_1^k\}$, $h_2 = \max_k\{h_2^k\}$ and $h_3 = \max_k\{h_3^k\}$.

Proof. Assuming that $\forall k = 1, \dots, m$; $(h_k)^2(h_3^k)^{-l-\frac{3}{2}d+\frac{3}{2}}$ and $(h_1^k)(h_2^k)^{-2l-3d+3}$ are sufficiently small, then Lemmas 3.2 and 3.4 are satisfied, so

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_\Lambda + \|\varphi - \varphi_h\|_\Phi \leq C\Big(\inf_{v_h \in V_h}\|u - v_h\|_V + \inf_{\mu_h \in \Lambda_h}\|\lambda - \mu_h\|_\Lambda + \inf_{\psi_h \in \Phi_h}\|\varphi - \psi_h\|_\Phi\Big)$$

and also for $\forall\ k = 1, \dots, m$ and when h_1^k is sufficiently small, one has

$$\|u - u_h\|_{H^1(\Omega_k)} \leq c_{k,1}(h_1^k)^{l-1}\|u\|_{H^l(\Omega_k)}.$$

Then

$$\begin{aligned}\|u - u_h\|_V^2 = \sum_k \|u - u_h\|^2_{H^1(\Omega_k)} &\leq C_1 \sum_k (h_1^k)^{2(l-1)}\|u\|^2_{H^l(\Omega_k)} \\ &\leq C_1(h_1)^{2(l-1)} \sum_k \|u\|^2_{H^l(\Omega_k)}\end{aligned} \tag{18}$$

where $C_1 = (\max_k(c_{k,1}))^2$ and $h_1 = \max_k\{h_1^k\}$. That can be written as

$$\|u - u_h\|_V^2 = \sum_k \|u - u_h\|^2_{H^1(\Omega_k)} \leq C_1(h_1)^{2(l-1)}\|u\|^2_{H^l(\Omega)}. \tag{19}$$

For h_2^k sufficiently small and $s+1<l$, we also have

$$\|\lambda^k-\lambda_h^k\|_{H^{-\frac{1}{2}}(\partial\Omega_k)} \le c_2(h_2^k)^{s+\frac{3}{2}}\|\lambda^k\|_{H^{s+1}(\partial\Omega_k)} \tag{20}$$

and

$$\begin{aligned}\|\lambda-\lambda_h\|_\Lambda^2 = \sum_k \|\lambda^k-\lambda_h^k\|^2_{H^{-\frac{1}{2}}(\partial\Omega_k)} &\le C_2\sum_k (h_2^k)^{2s+3}\|\lambda\|^2_{H^{s+1}(\partial\Omega_k)} \\ &\le C_2(h_2)^{2s+3}\sum_k \|\lambda\|^2_{H^{s+1}(\partial\Omega_k)}\end{aligned} \tag{21}$$

where $C_2=(\max_k\{c_{k,2}\})^2$ and $h_2=\max_k\{h_2^k\}$

$$\|\lambda-\lambda_h\|_\Lambda^2 \le C_2(h_2)^{2s+3}\|\lambda\|^2_{H^{s+1}(\partial\Omega)}. \tag{22}$$

In the other hand and as

$$\|.\|_\Phi \sim \sum_k |.|_{H^{\frac{1}{2}}(\partial\Omega_k)} \tag{23}$$

we have

$$\begin{aligned}\|\varphi-\varphi_h\|_\Phi &\le C_3\sum_k (h_3^k)^{s+\frac{3}{2}}|\varphi|_{H^{s+1}(\partial\Omega_k)} \\ &\le C_3(h_3)^{s+\frac{3}{2}}\sum_k |\varphi|_{H^{s+1}(\partial\Omega_k)}\end{aligned} \tag{24}$$

where $C_3=\max_k\{c_3^k\}$ and $h_3=\max_k\{h_3^k\}$.

The Eq. (24) can also be written as

$$\|\varphi-\varphi_h\|_\Phi \le C_3(h_3)^{s+\frac{3}{2}}\|\varphi\|_{H^{s+1}(\partial\Omega)}. \tag{25}$$

By using inequalities (19), (22) and (25) we have

$$\|u-u_h\|_V+\|\lambda-\lambda_h\|_\Lambda+\|\varphi-\varphi_h\|_\Phi \le C\left(h_1^{(l-1)}\|u\|_{H^l(\Omega)}+h_2^{s+\frac{3}{2}}\|\lambda\|_{H^{s+1}(\partial\Omega)}+h_3^{s+\frac{3}{2}}\|\varphi\|_{H^{s+1}(\partial\Omega)}\right). \tag{26}$$

So, the theorem is satisfied. □

4. Numerical results

Two different types of elliptic problems are treated herein. In the first simulation we assume that the function K is a constant equal to 1. Then we extend the three-field domain decomposition method to a problem with non-constant function K. In all numerical simulations, the Gauss method is used to approximate the integrals arising in the meshless mixed Galerkin formulation. Following [16], the radial basis function $\varphi(r)=r^5$ is introduced to construct the function spaces since it is proved in the same reference [16] that $\varphi(r)=r^5$ is the best function among the global radial basis functions, to give good numerical results. So, the function is then adopted in our work for the coupled meshless mixed Galerkin method with three-field domain decomposition method. Compactly supported radial basis functions are omitted because of the numerical integration difficulty over their supports and also because of the sensibility of their support parameter.

The errors used here are given by

$$L_{er}^\infty=\sup_\Omega\frac{|u_h-u|}{|u|} \quad\text{and}\quad L_{er}^2=\frac{\|u_h-u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}.$$

4.1. Example 1

The first problem to be treated is the following example given in [16].

$$\begin{cases}-\Delta u(x)=f & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases} \tag{27}$$

where $\Omega=[-1\ 1]\times[-1\ 1]$, $f(x,y)=4(x^2+y^2)\sin(x^2-1)\sin(x^2-1)-2\sin(x^2+y^2+2)$ and $u(x,y)=\sin(x^2-1)\sin(y^2-1)$ is the analytical solution of the problem.

Table 1
Influence of the number of Gauss points on the numerical solution (Example 1).

Gauss points	L^∞_{er}	L^2_{er}	Condition number
4	0.2912	1.615749e−001	4.5842e+018
6	0.0549	2.237044e−002	1.3634e+012
10	0.0423	2.329848e−002	2.6402e+012

Table 2
Results for example 1 using mixed Galerkin method.

$N \times M$	Numbers of unknowns	L^∞_{er}	L^2_{er}	Condition number	CPU
5×5	56	0.0423	2.329848e−002	2.6402e+012	0.7813s
6×6	73	0.0161	6.143874e−003	3.6059e+014	1.0469s
7×7	92	0.0132	5.037112e−003	1.1347e+015	1.5000s
8×8	113	0.0217	2.711726e−003	7.1563e+016	2.0781s
9×9	136	0.0035	2.013792e−003	3.2101e+017	2.9219s
10×10	161	31.2429	1.953537e+000	4.2557e+018	3.9688s

Table 3
Results for example 1 using coupled three-field decomposition with mixed Galerkin method.

Number of sub-domains	Numbers of unknowns	L^∞_{er}	L^2_{er}	Condition number	CPU
2	137	0.0200	6.0544e−005	1.3037e+014	1.9219s
4	321	0.0147	5.6811e−006	1.3734e+014	5.4531s
9	745	0.0161	2.0371e−006	1.9977e+014	24.7813s
9	976	0.0051	2.9378e−007	2.1135e+017	37.5469s

Table 4
Comparison of mixed Galerkin and Coupled method (Example 1).

Used method	Numbers of unknowns	L^∞_{er}	L^2_{er}	Condition number	CPU
Mixed Galerkin	136	0.0035	2.0137e−003	3.2101e+017	2.9219s
Coupled (2 sub-domains)	137	0.0200	6.0544e−005	1.3037e+014	1.9219s

Before applying the coupled method and comparing it to mixed Galerkin one used in [16], the influence of the integral approximation has been analyzed by solving the example using 25 nodes and increasing the number of integration Gauss points. It can be noticed from the Table 1 that using 10 Gauss points, best results are obtained for the errors and condition number.

To show the efficiency of the proposed technique, we first solve the problem using the meshless mixed Galerkin method and computing errors of the numerical solution obtained. Then, we solve the same problem by coupling the mixed Galerkin approximation with the three-field domain decomposition method. The number of nodes used over each sub-domain varies from 25 to 121. The results obtained are given in Tables 2 and 3 for the mixed Galerkin and the three-field domain decomposition methods, respectively.

Increasing the number of nodes from 25 to 81, we can notice from Table 2 that L^2_{er} and L^∞_{er} errors decrease from approximately 4.23×10^{-2} to 3.5×10^{-3} and from 2.32×10^{-2} to 2.01×10^{-3} respectively. But L^2_{er} and L^∞_{er} errors increase up to 34.24 and 1.95 respectively when using 100 nodes (See Table 2), which means that a robust technique is needed.

The comparison of these results with those in Table 3 shows that the application of the three-field domain decomposition approximation reduces the L^2_{er} error radically down to 2.0371×10^{-6} when using 9 sub-domains with 745 unknowns. The performance of three-field domain decomposition is demonstrated by remarking that the numerical solution obtained by the application of three-field domain decomposition, using 2 sub-domains and 137 unknowns, is more accurate than the one with 136 unknowns, applying mixed Galerkin method only (See Table 4). One of the advantages of the coupled three-field method and radial basis functions over the mixed Galerkin method is that the condition number is kept constant (around 10^{14} for this example, see Table 3) through all simulation tests, although the number of sub-domains and so the number of nodes are greatly increased, which is not the case for mixed Galerkin method only. From Table 2, we can observe that the condition number is increasing with respect to the number of nodes used.

We can also observe from Table 3 that the L^∞_{er} error is mildly increased from 0.0147 using 4 sub-domains into 0.0161 using 9 sub-domains. To decrease it we just increase the number of nodes to have 976 unknowns and keep the number of sub-domains to 9, and then we observe, from Table 3, that the L^∞_{er} is decreased to 0.0051 but the condition number is increased to 2.1135×10^{17}. We conclude that the balance between errors and condition number is very sensitive.

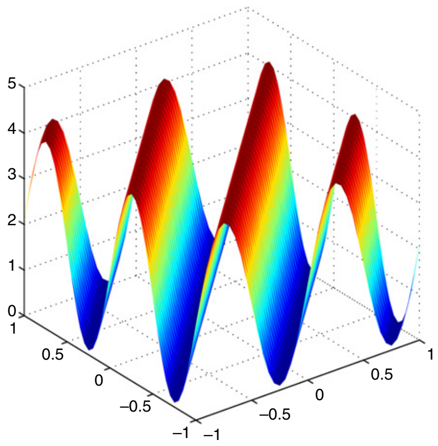

Fig. 1. The function K with $\alpha = 2.1$; $\beta = 2$ and $\lambda = 2$ (Example 2).

Table 5
Results for example 2 using mixed Galerkin.

$N \times M$	Numbers of unknowns	L^∞_{er}	L^2_{er}	Condition number	CPU
4×4	41	0.0428	2.242405e−002	1.1333e+011	0.9375s
5×5	56	0.0942	5.292274e−002	2.9011e+012	1.2969s
6×6	73	0.1493	7.630371e−002	3.6281e+014	1.6250s
8×8	113	0.6571	4.180773e−001	6.9679e+016	2.9688s
10×10	161	76.3678	1.967796e+001	2.3249e+016	5.3750s

Table 6
Results for example 2 using coupled three-field decomposition with mixed Galerkin method.

Number of sub-domains	Numbers of unknowns	L^∞_{er}	L^2_{er}	Condition number	CPU
2	147	0.1517	0.0014	2.1737e+014	2.5938s
4	317	0.0136	4.9239e−005	4.4033e+014	5.6875s
9	733	0.0283	5.0320e−006	9.2990e+014	23.8750s

4.2. Example 2

The second example treated is a problem with the following input function $K(x, y) = (\alpha + \beta \sin(\lambda\pi(x - y)))$, where $\alpha = 2.1, \beta = 2, \lambda = 2$ and $\Omega = [-2\ 2] \times [-2\ 2]$. The Fig. 1 shows the behavior of the function K. The example is an elliptic boundary value problem with oscillating coefficient that needs a sophisticated numerical method to solve [18].

We choose the second right hand side function f to be

$$f(x, y) = 2[\alpha + \beta * \sin(\lambda\pi(x - y))][(y^2 - 4) + (x^2 - 4)] + \beta\lambda\pi * \cos(\lambda\pi(x - y))[2x(y^2 - 4) - 2y(x^2 - 4)]$$

so that the analytical solution of the problem is $u(x, y) = (x^2 - 4)(y^2 - 4)$.

The example is treated in the same way as it was done for the first one which means that we first apply the mixed Galerkin method and then the coupled technique. Tables 5 and 6 show the errors of the applied methods with respect to number of nodes for mixed Galerkin approximation and with respect to number of sub-domains for coupled three-field domain decomposition technique. Good results are obtained with 9 sub-domains. As in the first example the condition number is still approximately constant and around 10^{14}. The small increase of L^∞_{er} error for 9 sub-domains can be caused by the existing corners points or/and roundoff errors.

From Table 5 we notice that with 113 unknowns, the errors increase and bad accuracy is shown using mixed Galerkin method. But regarding Table 6 we remark that using the coupled technique we still get results with more than 700 nodes.

4.3. Example 3

Here, we use the same example (as example 2) but with different parameters: α, β and λ, which are taken to be 2.1, 2 and 4, respectively. With these input data the function K is highly oscillating. Its behavior is given in Fig. 2.

Analyzing the results obtained in Table 7 of example 3 we can observe that a bad approximated solution is obtained. The errors are increased with respect to number of nodes. The same results can be seen in Fig. 3. These results are in concordance

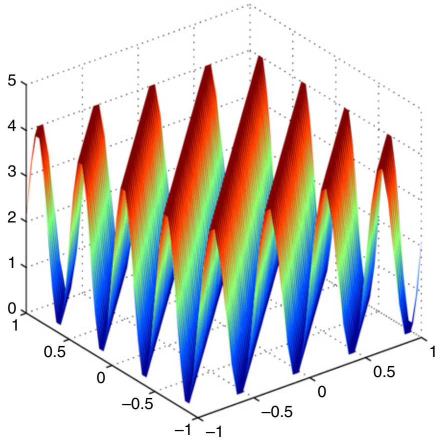

Fig. 2. The function K with $\alpha = 2.1$, $\beta = 2$ and $\lambda = 4$ (Example 3).

Table 7
Results for example 3, using mixed Galerkin method without decomposition.

$N \times M$	Numbers of unknowns	L^{∞}_{er}	L^{2}_{er}	Condition number
4×4	41	1.5424	9.849152e−001	1.0644e+011
5×5	56	2.7562	1.370275e+000	2.5793e+012
6×6	73	5.0459	2.090744e+000	3.4432e+014
8×8	113	5.2360	2.384145e+000	6.2556e+016

Table 8
Results using coupled three-field domain decomposition with mixed Galerkin method.

Number of sub-domains	Numbers of unknowns	L^{∞}_{er}	L^{2}_{er}	Condition number
2	169	0.1000	0.0035	1.9280e+015
4	317	0.0299	3.1893e−004	9.6886e+015
9	793	0.0208	2.6342e−004	1.9294e+016
16	1225	0.0098	1.9242e−005	6.6531e+016

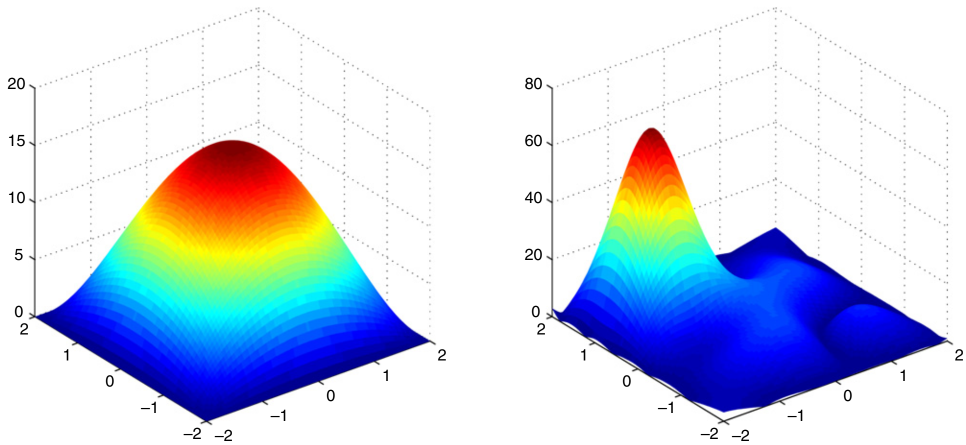

Fig. 3. The exact on the left and numerical solution using direct MGRBF without domain decomposition on the right (Example 3).

with the behavior of the input function K which is highly oscillating which classify the problem among a multiscale boundary value problems. The numerical treatment of this kind of problem needs more discretizing nodes or a very sophisticated method [18]. What is surprising is that good results are obtained when coupling the mixed Galerkin method with three-field domain decomposition (see Table 8 and Figs. 4–7). It is shown that the accuracy and the stability of the numerical approximation are less influenced by the oscillation of the function K. Good results are obtained using 16 sub-domains (Fig. 7) which generates 1225 unknowns Fig. 5.

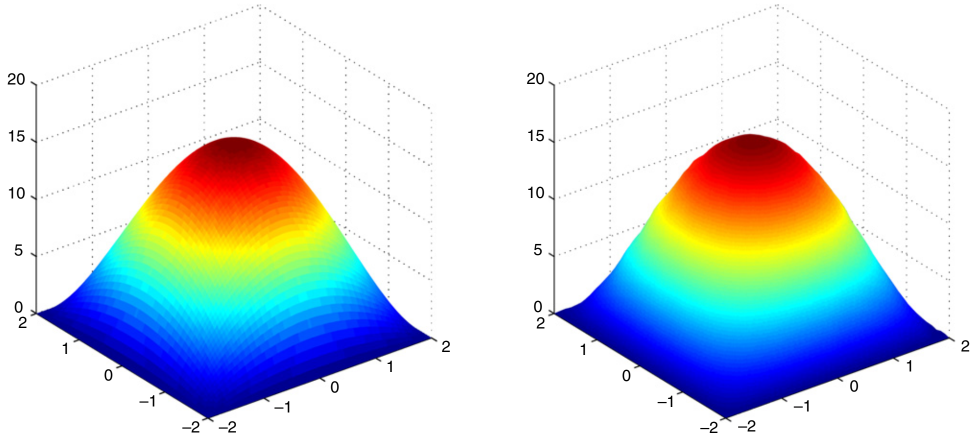

Fig. 4. Exact solution on the left and numerical solution using 9 sub-domains on the right (Example 3).

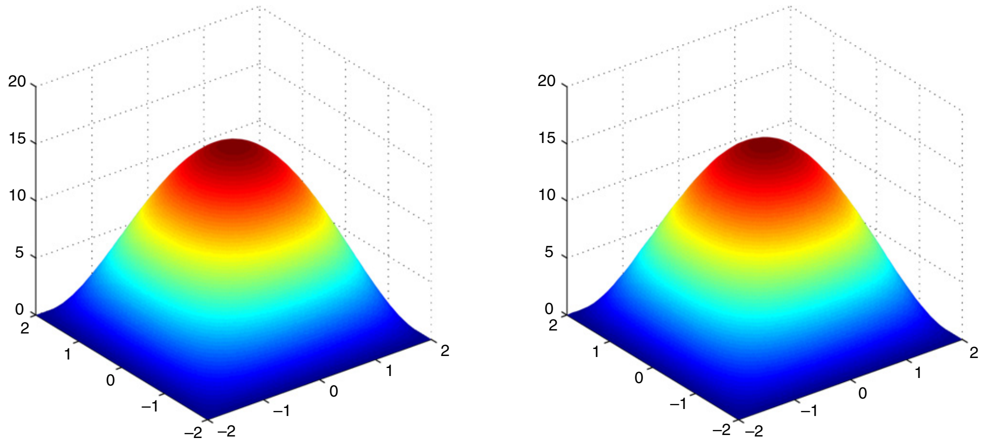

Fig. 5. Exact solution on the left and numerical solution using 16 sub-domains on the right (Example 3).

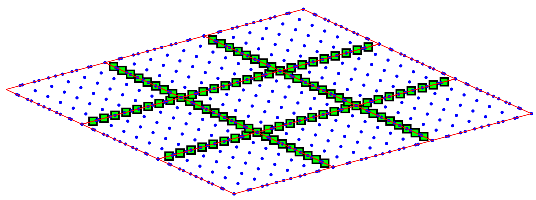

Fig. 6. Distributed center and 9 sub-domains.

5. Conclusion

In this work, it has shown that using radial basis function and three-field domain decomposition the condition number has been successfully stabilized and a better accuracy of the solution was obtained. The numerical results obtained show the performance of the technique and its efficiency. It also shows that not only the increase of number of nodes can decrease the errors but also their distributions. So, a suitable technique for selecting the best distribution of nodes is necessary to the performance of the coupled method. The coupled technique is suitable for using parallel computation and solving multiscale problems since Bertoluzza has mentioned in [11] that "one of the interests of the three-field decomposition method lies in the observation that, for a given $\phi \in \Phi$, the computation of u and λ reduces to solving m independent Dirichlet problems on the sub-domain Ω_k for $k = 1, \ldots, m$". Further work will focus on the application of the technique to realistic problems and also on solving multiscale problems by the coupled technique in parallel.

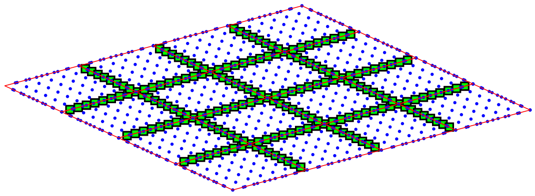

Fig. 7. Distributed center and 16 sub-domains.

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